B.Sc. (Math) part III

paper – V

Topic: Darboux Condition of Integrability

Theorem Let f be a bounded function on a closed bounded interval $[a,b]$ then to every $\varepsilon > 0$ there corresponds $\delta > 0$ such that

$$U(P) < \overline{\int_a^b} f(x)dx + \varepsilon \text{ and}$$

$$L(P) > \underline{\int_a^b} f(x)dx - \varepsilon$$

for all partitions P of (a,b) with norm $\mu(P) < \delta$

Lemma:- Let $|f(x)| \le K$ for $x \in [a,b]$ Let δ be a +ve number and let P_1 be partition of $[a,b]$ such that $\mu(P_1) \le \delta$

Let P_2 be a partition of $[a,b]$ consisting of all the points of P_1 and at the most some p more then we have

$$U(P_1) - 2pK\delta \le U(P_2) \le U(P_1)$$

proof of Lemma:– First suppose

the $p=1$ so that only one interval $\delta_r' \quad \delta_r''$ say δ_r, of P_1 is divided into two subintervals δ_r

δ_r' and δ_r'' Let M_r, M_r', M_r'' be the lub of f in $\delta_r, \delta_r', \delta_r''$ respectively we have

$$U(P_1) - U(P_2) = M_r|\delta_r| - (M_r'|\delta_r'| + M_r''|\delta_r''|)$$

$$= (M_r - M_r')\cdot|\delta_r'| + (M_r - M_r'')|\delta_r''|$$

for $|\delta_r| = |\delta_r'| + |\delta_r''|$

Now since $|f(x)| \leq K$ for all $x \in [a,b]$ therefore $-K \leq M_r' \leq M_r \leq K$

i.e $0 \leq M_r - M_r' \leq 2K$

Similarly $0 \leq M_r - M_r'' \leq 2K$

$\therefore 0 \leq U(P_1) - U(P_2) \leq 2K(|\delta_r'| + |\delta_r''|)$

$= 2K|\delta_r| \leq 2K\delta$

Now suppossing that each ~~additional~~ additional point is introduced one by one we obtain the result

we have prove the Main theorem

$\therefore$ As f is bounded $\exists K > 0$ Such that $|f(x)| \leq K$ for all $x \in [a,b]$

since $\int_a^b f(x)\,dx = \text{glb}(U(P))$

$\exists$ a partition $P_1 = \{a = x_0, x_1, \ldots x_p = b\}$ such that

$$U(P_1) < \overline{\int_a^b} f(x)\,dx + \frac{\epsilon}{2}$$

The poin[t]s P_1 are $(p+1)$ in number

Let δ be the +ve number such that

$$2k(p-1)\cdot\delta = \frac{\epsilon}{2}$$

Let P be any partition of $[a,b]$ with $\mu(P) < \delta$

Let P_2 be the common refinement of P_1 and P.

From the above lemma we have

$$U(P) - 2(p-1)k\delta \le U(P_2) \le U(P)$$

Also $U(P_2) \le U(P_1)$

Thus we obtain

$$U(P) - 2(p-1)k\delta \le U(P_1)$$

$$\therefore U(P) \le 2(p-1)k\delta + U(P_1)$$

$$< \frac{\epsilon}{2} + \overline{\int_a^b} f(x)\,dx + \frac{\epsilon}{2}$$

$$= \overline{\int_a^b} f(x)\,dx + \epsilon$$

Similarly it can be proved that

$$L(P) > \underline{\int_a^b} f(x)\,dx - \epsilon$$

for all partition P with $\mu(P) \le \delta$